Student's Name: ______________________

Student Number: | | | | | | | | | |

Teacher's Name: ______________________

ABBOTSLEIGH

2024

HIGHER SCHOOL CERTIFICATE
Assessment 4

Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen.
- **NESA approved** calculators may be used.
- **NESA approved** reference sheet is provided.
- All necessary working should be shown in every question.
- Make sure your HSC Candidate Number is on the front cover of each booklet.
- Start a new booklet for each Question.
- Answer the Multiple Choice questions on the answer sheet provided.
- If you do not attempt a whole question, you must still hand in the Writing Booklet, with the words **'NOT ATTEMPTED'** written clearly on the front cover.

Total marks - 70

- Attempt Sections I and II.

Section I Pages 3 - 7

10 marks
- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II Pages 8 - 18

60 marks

- Attempt Questions 11 – 14.
- Allow about 1 hour and 45 minutes for this section.
- All questions are of equal value.

Outcomes to be assessed:

Year 11 Mathematics Extension 1 outcomes

A student:

ME11-1

uses algebraic and graphical concepts in the modelling and solving of problems involving functions and their inverses

ME11-2

manipulates algebraic expressions and graphical functions to solve problems

ME11-3

applies concepts and techniques of inverse trigonometric functions and simplifying expressions involving compound angles in the solution of problems

ME11-4

applies understanding of the concept of a derivative in the solution of problems, including rates of change, exponential growth and decay and related rates of change

ME11-5

uses concepts of permutations and combinations to solve problems involving counting or ordering

ME11-7

communicates making comprehensive use of mathematical language, notation, diagrams and graphs

Year 12 Mathematics Extension 1 outcomes

A student:

ME12-1

applies techniques involving proof or calculus to model and solve problems

ME12-2

applies concepts and techniques involving vectors and projectiles to solve problems

ME12-3

applies advanced concepts and techniques in simplifying expressions involving compound angles and solving trigonometric equations

ME12-4

uses calculus in the solution of applied problems, including differential equations and volumes of solids of revolution

ME12-7

evaluates and justifies conclusions, communicating a position clearly in appropriate mathematical forms

Section I (10 marks)

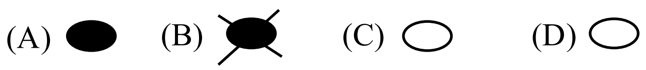
Attempt Questions 1 – 10

Use the multiple-choice answer sheet

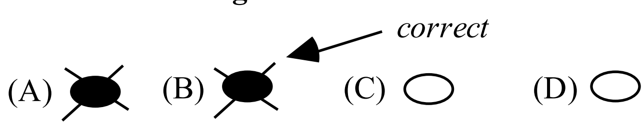
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.



If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.



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1. The random variable X represents the number of successes in 12 independent Bernoulli trials. The probability of success is $p = 0.7$ in each trial.

   Which of the following statements correctly state the mean and variance of X?

   A. The mean is 8.4 and the variance is 1.59

   B. The mean is 8.4 and the variance is 2.52

   C. The mean is 12 and the variance is 1.59

   D. The mean is 12 and the variance is 2.52

2 Which of the following vectors are perpendicular?

A. $\underset{\sim}{u} = 3\underset{\sim}{i} - 7\underset{\sim}{j}$, $\underset{\sim}{v} = \frac{7}{3}\underset{\sim}{i} - \underset{\sim}{j}$

B. $\underset{\sim}{u} = 14\underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{u} = \frac{1}{2}\underset{\sim}{i} - 7\underset{\sim}{j}$

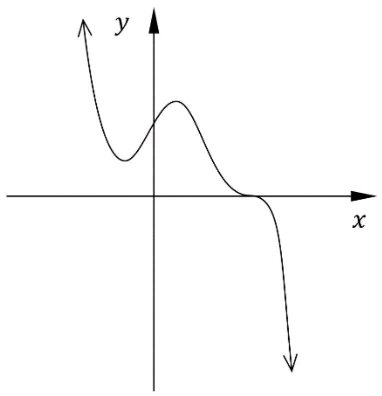
C. $\underset{\sim}{u} = \frac{9}{2}\underset{\sim}{i} + \frac{7}{2}\underset{\sim}{j}$, $\underset{\sim}{u} = -\frac{2}{9}\underset{\sim}{i} - \frac{1}{7}\underset{\sim}{j}$

D. $\underset{\sim}{u} = \frac{2}{5}\underset{\sim}{i} - 2\underset{\sim}{j}$, $\underset{\sim}{u} = 5\underset{\sim}{i} + 3\underset{\sim}{j}$

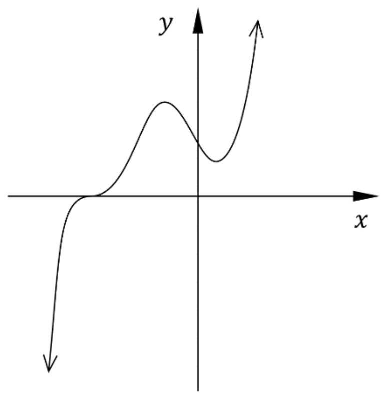
3 A monic polynomial of degree 5 has one repeated root of multiplicity 3, and is divisible by $x^2 - x$.

Which of the following could be the graph of $y = P(x)$?

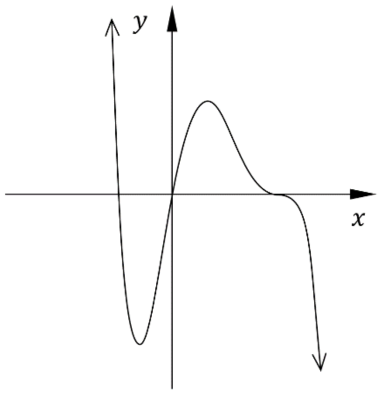
A.



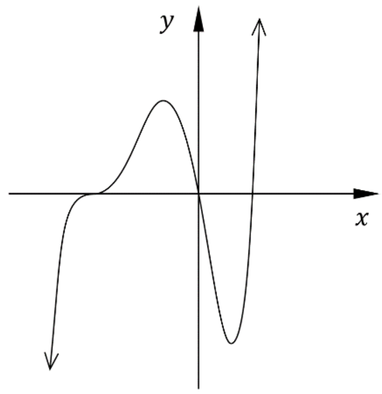
B.



C.



D.



4 The coefficient of the fourth term in the expansion of $(3x-4)^6$ is

A. $-34\ 560$

B. $34\ 560$

C. $25\ 920$

D. $-25\ 920$

5 If $f(x) = \dfrac{4x}{x-2}$, what is the domain of $f^{-1}(x)$?

A. $x \in [4, \infty)$

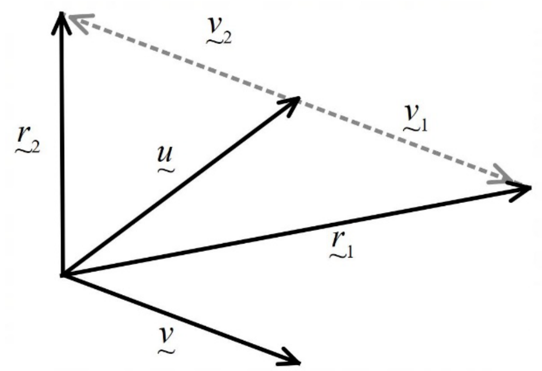
B. $x \in (-\infty, 2) \cup (2, \infty)$

C. $x \in (-\infty, 4) \cup (4, \infty)$

D. $x \in (-\infty, \infty)$

6 The diagram shows two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Two resultant vectors, $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$, are constructed using $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ which are parallel to, and equal in length to $\underset{\sim}{v}$.



NOT TO SCALE

Which statement is true?

A. $\underset{\sim}{r}_1 = \underset{\sim}{u} - \underset{\sim}{v}$ and $\underset{\sim}{r}_2 = \underset{\sim}{u} + \underset{\sim}{v}$

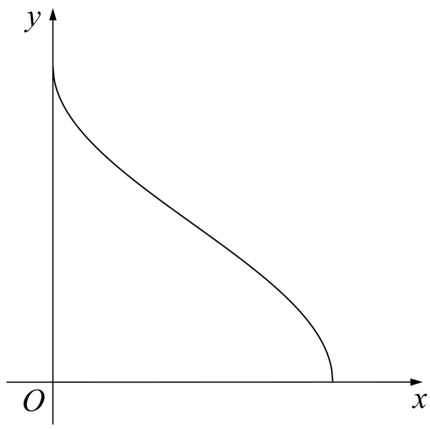
B. $\underset{\sim}{r}_1 = \underset{\sim}{v} + \underset{\sim}{u}$ and $\underset{\sim}{r}_2 = \underset{\sim}{v} - \underset{\sim}{u}$

C. $\underset{\sim}{r}_1 = \underset{\sim}{v} - \underset{\sim}{u}$ and $\underset{\sim}{r}_2 = \underset{\sim}{u} - \underset{\sim}{v}$

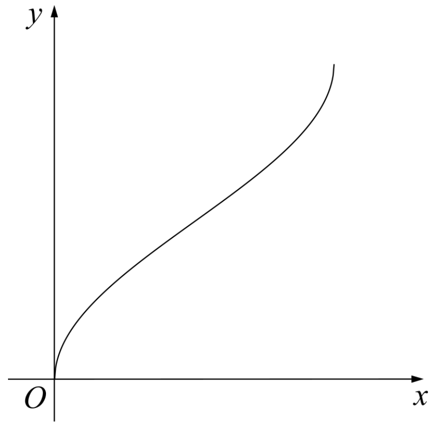
D. $\underset{\sim}{r}_1 = \underset{\sim}{u} + \underset{\sim}{v}$ and $\underset{\sim}{r}_2 = \underset{\sim}{u} - \underset{\sim}{v}$

7 Which diagram best represents the graph of $y = 2\sin^{-1}\left(1 - \frac{x}{3}\right)$?

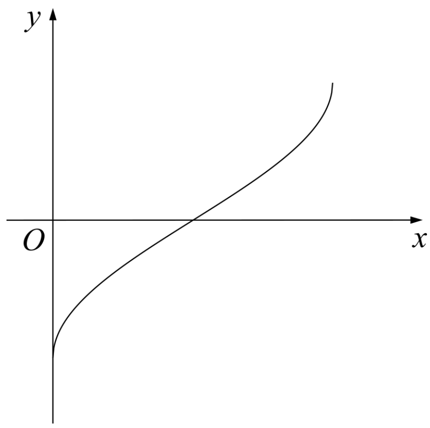
A.



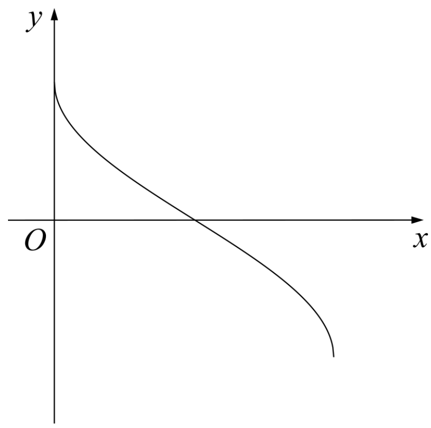
B.



C.



D.



8 For the polynomial $P(x) = x^3 - 3x^2 + 4$, it is known that $P'(2) = 0$.

Which of the following statements is **incorrect**?

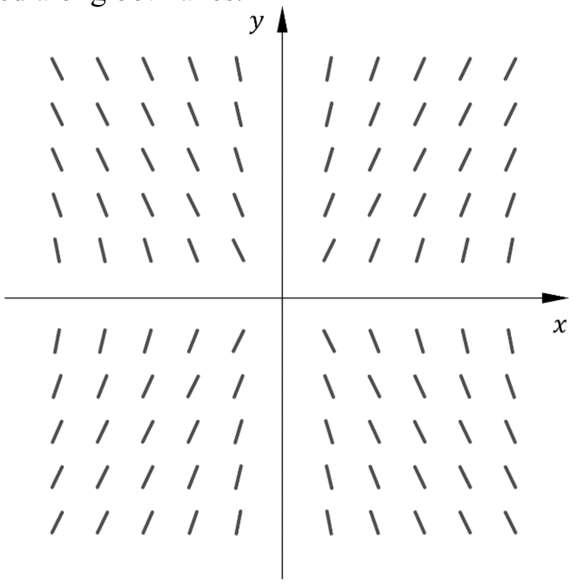
A. $P(2) = 0$.

B. $P(x)$ has one of its roots at $x = -1$.

C. $P(x)$ has a double root at $x = -2$.

D. $P(x)$ is divisible by $(x - 2)$.

9 The diagram shows the direction field of a differential equation. The differential equation is undefined along both axes.



Which of the following differential equations is best represented by the direction field above?

A. $\dfrac{dy}{dx}=\dfrac{xy}{x+y}$

B. $\dfrac{dy}{dx}=\dfrac{x}{y}+\dfrac{y}{x}$

C. $\dfrac{dy}{dx}=\dfrac{x+y}{xy}$

D. $\dfrac{dy}{dx}=\dfrac{x}{y}-\dfrac{y}{x}$

10 A body of still water has suffered an oil spill and a circular oil slick is floating on the surface of the water. The area of the oil slick is increasing by $0.1\ \text{m}^2$/minute.

At what rate is the radius increasing when the area is $0.3\ \text{m}^2$?

A. 0.0515 m/min

B. 0.0098 m/min

C. 0.03 m/min

D. 0.0531 m/min

End of Section I

SECTION II
60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a **SEPARATE** writing booklet. Extra writing booklets are available.

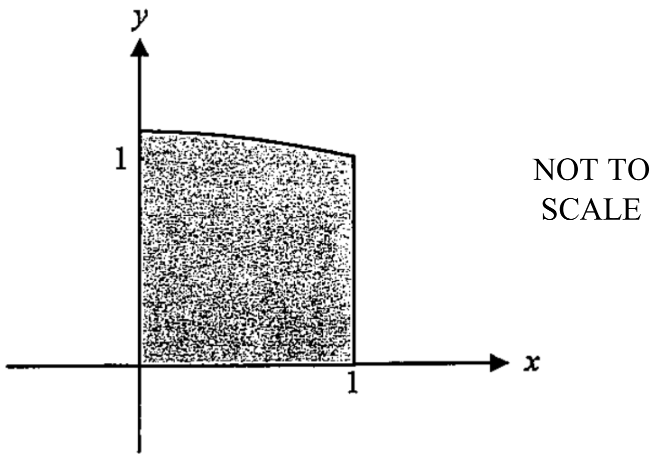
In Questions 11-14 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve the inequality, $\dfrac{2-3x}{7x+2} \le -2$. **3**

(b) Find $\displaystyle\int_0^2 2x\sqrt{1-\frac{x}{2}}\,dx$, using the substitution $u = 1-\dfrac{x}{2}$. **3**

(c) **3**



The region bounded by the function $y = \dfrac{2}{\sqrt{x^2+3}}$, the x-axis, the y-axis and the line $x = 1$, is shaded in the diagram above. Find the exact volume when this region is rotated about the x axis.

Question 11 continued on next page

(d) In a particular racing car, the probability that the brake pads will need replacing during a race is 0.2. The car competes in a 12 race series.

(i) What is the probability that the pads will need to be replaced on exactly 3 occasions? Give your answer correct to 2 decimal places. **1**

(ii) What is the probability that the pads will need to be replaced on 3 occasions at most? Give your answer correct to 2 decimal places. **2**

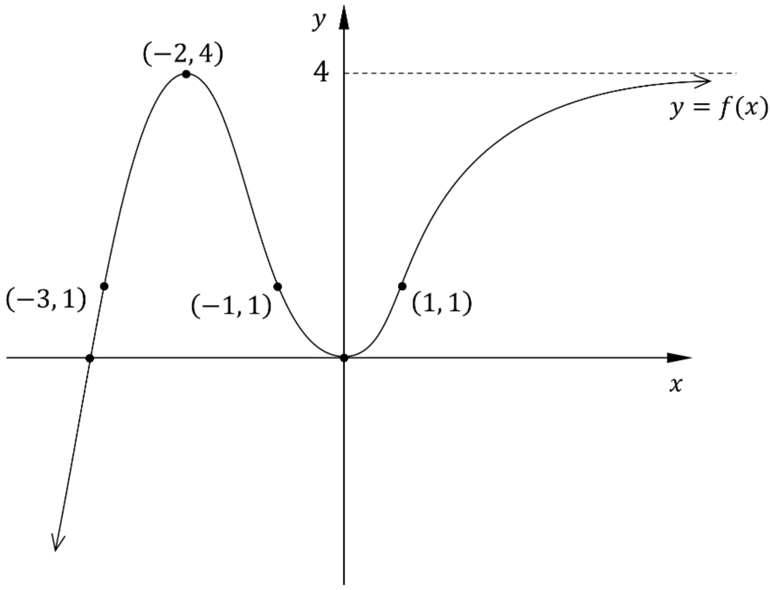
(e) Evaluate $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-4x^2}}$. **2**

(f) What is the minimum number of students required in a class to ensure that three of them are born in the same month? **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram of $y = f(x)$ is shown below. **2**



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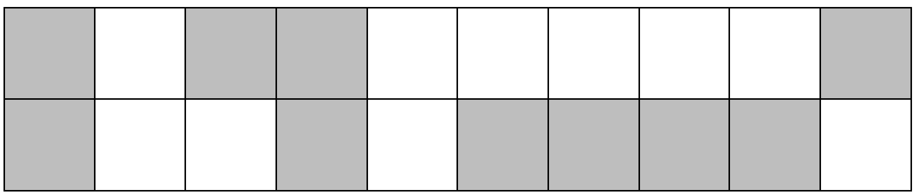
On the diagram provided on page 1 of the question 12 writing booklet, sketch the graph of $y = \sqrt{f(x)}$, clearly showing all turning points, intercepts, asymptotes and points of intersection with $y = f(x)$.

(b) Find the particular solution to the differential equation $\frac{dy}{dx} = 2xy$, given that the graph passes through (0,2). **3**

(c) You are given two rows of square tiles, 10 tiles in each row. You are asked to shade 4 tiles in the top row and 6 tiles in the bottom row so that exactly two shaded tiles in the top row are directly above a shaded tile in the bottom row, An example of one such arrangement is shown. **2**



In how many ways is this possible?

Question 12 continued on next page

(d) $\sqrt{6}\sin x+\sqrt{2}\cos x$ can be expressed in the form $R\sin(x+\alpha)$, where $R>0$ and $0<\alpha<\frac{\pi}{2}$. **3**

By using this result or otherwise solve

$$\sqrt{6}\sin x+\sqrt{2}\cos x=2 \qquad x\in[0,2\pi].$$

(e) Use mathematical induction to prove that 9^n-4^n is divisible by 5 for all integers $n\geq 1$. **3**

(f) Given $\underset{\sim}{u}=2\underset{\sim}{i}+3\underset{\sim}{j}$ and $\underset{\sim}{v}=-2\underset{\sim}{i}+4\underset{\sim}{j}$, find $proj_{\underset{\sim}{u}}\,\underset{\sim}{v}$. **2**

End of Question 12

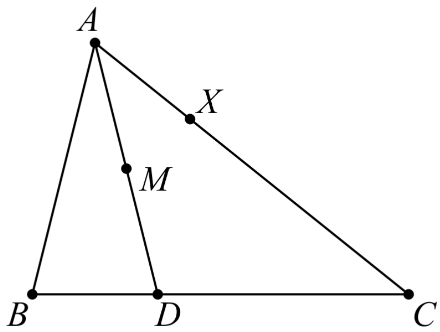
Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Prove that $\tan 4\theta - \tan\theta = \dfrac{\sin 3\theta}{\cos 4\theta \cos\theta}$. **2**

(ii) Hence solve $\tan 4\theta = \tan\theta$ for $0 \le \theta \le \pi$. **2**

(b) The diagram below shows a triangle ABC whose vertices have position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ from an origin O.

Point D lies on BC such that $BD = \dfrac{1}{3}BC$, point X lies on AC such that $AX = \dfrac{1}{4}AC$ and M is the midpoint of AD.



NOT TO SCALE

(i) Express the vector AM in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$. **3**

(ii) Show that B, M and X are collinear, hence find $BM : MX$. **3**

Question 13 continued on next page

Question 13 (continued)

(c) After time t years from the start of the year 2021, the number of people in a population is given by $N = 70 + Ae^{(0\cdot 1t)}$ where A is a constant greater than zero.

(i) Show that $\frac{dN}{dt} = 0.1(N - 70)$. **1**

(ii) There were 100 people in the population at the start of the year 2021. Find the year when the population size will exceed 190. **2**

(d) Show that the normal to the curve $y = \sin^{-1}\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}$ at the point where $x = -\frac{\pi}{2}$ passes through the origin. **2**

End of Question 13

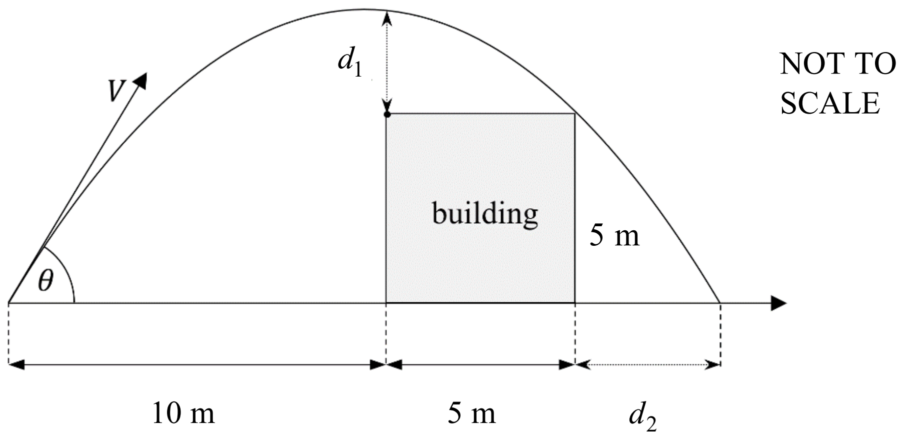
Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A student kicks a football from ground level. The ball is kicked at an angle of projection of θ at a speed of V metres per second. **5**

The displacement vector of the ball from the student's position, t seconds after it is kicked, is given by

$$r(t) = \left(Vt\cos\theta\right)\underset{\sim}{i} + \left(Vt\sin\theta - 5t^2\right)\underset{\sim}{j}\ .$$

Two seconds after being kicked, the ball just clears the far side of a 5 metre high building. The building is 5 metres wide, and the closest wall of the building is 10 metres from the student.



Let d_1 be the distance by which it clears the closest wall of the building, and d_2 be the distance from the far wall of the building to the point of impact, as shown.

Show that the difference between d_2 and d_1 is less than 1. (ie: $d_2 - d_1 < 1$).

Question 14 continues on next page

Question 14 (continued)

(b) A team of biologists released 500 fish into a lake with a maximum carrying capacity of 10 000 fish. The fish population, N, in the lake after t years is modelled by the logistic equation

$$\frac{dN}{dt} = kN(10\,000 - N) \text{ where } k \text{ is a constant.}$$

(i) Given $\dfrac{10\,000}{N(10\,000-N)} = \dfrac{1}{N} + \dfrac{1}{(10\,000-N)}$, solve the differential equation to **3**

show that the fish population that t years is $N = \dfrac{10\,000}{1+19e^{-10000kt}}$.

(ii) It was found that the number of fish tripled during the first year, hence **2**

$$k = \frac{1}{10\,000}\ln\left(\frac{57}{17}\right)$$

Find correct to the nearest month, how long will it take for the fish population in the lake to reach 7000 fish?

Question 14 continues on next page

(c) *You may use the information on page 17 to answer this question.*

A binomial variable, X, is known to have a probability of success, p, of close to 0.81, but the exact value is unknown.

(i) A mathematician needs to estimate the value of p to within one-thousandth of the true result, with 99.7% certainty. **3**

To find a more accurate estimate of p, the mathematician plans to perform n trials, and find $\hat{p}$.

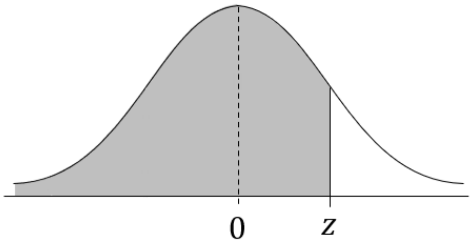
What is the minimum number of trials the mathematician needs to perform so that $\hat{p}$ is accurate to within one-thousandth of the true value of p, with 99.7% certainty?

(ii) The mathematician repeats the experiment 2 million times, recording 1 615 580 successes. What is the probability that in 10 000 trials the number of successes would be greater than 8 100, correct as a percentage to 1 decimal place? **2**

End of Paper

You may use the information below to answer Question 14 (c) (ii).

Table of values $P(Z < z)$ for the normal distribution $N(0, 1)$

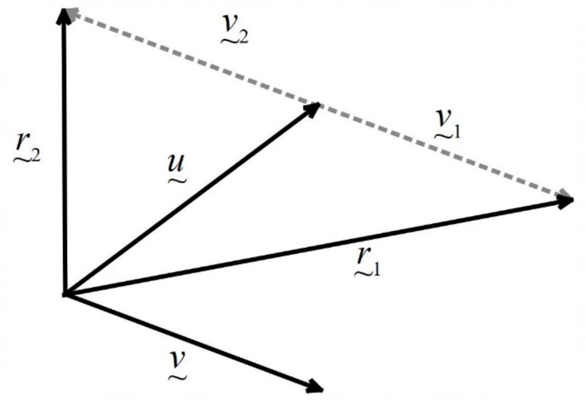


Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.10	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.20	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.30	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.40	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.50	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.60	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.70	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.80	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.90	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.00	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.10	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.20	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.30	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.40	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.50	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.60	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.70	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.80	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.90	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.00	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.10	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.20	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.30	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.40	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.50	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.60	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.70	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.80	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.90	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.00	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Extension 1 Task 4 2024 Solutions

Section I

1.	$n = 12,\ p = 0.7$ $\mathrm{E}(X) = \bar{x}$ $= np$ $= 12 \times 0.7$ $= 8.4$ $\mathrm{Var}(X) = np(1-p)$ $= 12 \times 0.7 \times 0.3$ $= 2.52$	**B**
2.	The perpendicular vectors have a zero dot product. The option D is correct because: $\underset{\sim}{u} \cdot \underset{\sim}{v} = \left(14\,\underset{\sim}{i} + \underset{\sim}{j}\right) \cdot \left(\frac{1}{2}\underset{\sim}{i} - 7\,\underset{\sim}{j}\right)$ $= \left(14 \cdot \frac{1}{2}\right) + (1 \cdot -7)$ $= 7 - 7 = 0$	**B**
3.	Options A and C can be eliminated since a monic quintic starts bottom left and finishes top right. $x^2 - x = x(x-1)$, so the polynomial is also divisible by x and $x - 1$, so the curve cuts through the x-axis at the origin and $(1,0)$, and it has a triple root (which could be at the origin or $(1,0)$, so D.	**D**
4.	$(3x-4)^6 = \sum_{r=0}^{6} \binom{6}{r} (3x)^{6-r}(-4)^r$. 4th term is $\binom{6}{3}(3x)^3(-4)^3$. Coefficient of 4th term is $\frac{6!}{3!3!} \times 3^3 \times (-4)^3 = -34560$	**A**
5.	Note that $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{4x}{x-2} = \lim_{x\to\pm\infty} \frac{4}{1-\frac{2}{x}} = 4$. So, the horizontal asymptote is $y = 4$, and the range of f is $\mathbb{R}/\{2\}$. Since the domain of the inverse function f^{-1} is the range of the function f, so the solution is $(-\infty, 4) \cup (4, \infty)$	**C**

6.	 The addition of vectors is represented geometrically by translating the vectors top to tail, so $\underset{\sim}{v}$ is translated to $\underset{\sim}{v}_1$, giving the resultant $\underset{\sim}{r}_1$ and subtraction by reversing the direction of $\underset{\sim}{v}$ and translating it to $\underset{\sim}{v}_2$ giving the resultant $\underset{\sim}{r}_2$. So $\underset{\sim}{r}_1 = \underset{\sim}{u} + \underset{\sim}{v}$ and $\underset{\sim}{r}_2 = \underset{\sim}{u} - \underset{\sim}{v}$.	**D**
7.	domain: $-1 \le 1 - \frac{x}{3} \le 1$ $-2 \le -\frac{x}{3} \le 0$ $0 \le x \le 6$ range: $-\frac{\pi}{2} \le \sin^{-1}\left(1 - \frac{x}{3}\right) \le \frac{\pi}{2}$ $-\pi \le 2\sin^{-1}\left(1 - \frac{x}{3}\right) \le \pi$	**D**
8.	$P(x) = x^3 - 3x^2 + 4 = (x-2)^2(x+1)$ $\therefore$ double root at $x = 2$ and is divisible by $(x-2)$ single root at $x = -1$ $P(2) = 0$ Statements A, B and D are all true. C is incorrect as the double root is at x=2, not x=-2	**C**
9.	The direction field is undefined along both axes, so the denominator cannot be zero when one or both of x or y is zero, so eliminate A as it would otherwise show horizontal lines along each axis except at the origin. We can eliminate C as when $y = -x$ the numerator would be zero, so we would see horizontal slopes along the line $y = -x$, which do not appear. We can eliminate D, as when $y = \pm x$ the gradient would be zero, so we would see horizontal slopes along the lines $y = \pm x$, which do not appear. We can confirm B, as $\frac{x}{y} + \frac{y}{x} = \frac{x^2+y^2}{xy}$. The numerator is always positive, while the denominator is positive when x and y have the same sign (1st and 3rd quadrants) giving positive slopes, and negative when they are of opposite sign (2nd and 4th quadrants) giving negative slopes.	**B**

10.	$A = \pi r^2$ $0.3 = \pi \times r^2 \qquad \therefore r = \sqrt{\frac{0.3}{\pi}}$ $\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$ $= \frac{1}{2\pi r} \times 0.1$ $= \frac{1}{2\pi \times \sqrt{\frac{0.3}{\pi}}} \times 0.1$ $= 0.0515$	**A**

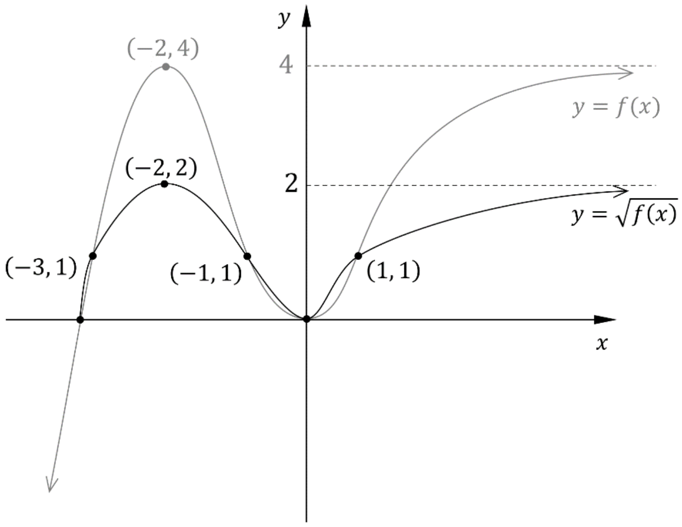
Section II

Question 11 (15 marks) **Marks**

(a)	$\frac{2-3x}{7x+2} \le -2.$ $x \ne -\frac{2}{7}$ ☑ $\frac{2-3x}{7x+2} \le -2.\ \times(7x+2)^2$ to both sides $(2-3x)(7x+2) \le -2(7x+2)^2$ $(2-3x)(7x+2)+2(7x+2)^2 \le 0$ $(7x+2)(2-3x+2(7x+2)) \le 0$ $(7x+2)(11x+6) \le 0$ ☑ $-\frac{6}{11} \le x < -\frac{2}{7}$ ☑	**3**
(b)	$\int_0^2 2x\sqrt{1-\frac{x}{2}}\,dx$ | let $u=1-\frac{x}{2}$ so $\frac{du}{dx}=-\frac{1}{2}$ | If $x=2$, $u=0$ $=\int_1^0 -4(u-1)u^{\frac{1}{2}}\times -2\frac{du}{dx}dx$ | So $x=-2(u-1)$ | and if $x=0$, $u=1$ $=8\int_1^0\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du$ $=8\left[\frac{2u^{\frac{5}{2}}}{5}-\frac{2}{3}u^{\frac{3}{2}}\right]_1^0$ $=8\left\{(0-0)-\left(\frac{2}{5}-\frac{2}{3}\right)\right\}$ $=\frac{32}{15}$ ☑ changing limits into u ☑ substitution of u ☑ answer	**3**

(c)	$\text{Volume required} = \pi\int_0^1 \frac{4}{x^2+3}dx$ ☑ $= \frac{4\pi}{\sqrt{3}}\int_0^1 \frac{\sqrt{3}}{x^2+3}dx$ $= \frac{4\pi}{\sqrt{3}}\left[\tan^{-1}\frac{x}{\sqrt{3}}\right]_0^1$ ☑ $= \frac{4\pi}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0\right)$ $= \frac{4\pi}{\sqrt{3}}\times\frac{\pi}{6}$ $= \frac{2\sqrt{3}\pi^2}{9}$ cubic units. ☑	**3**
(d)(i)	$= {}^{12}C_3(0.2)^3(0.8)^9$ $= 0.23622..$ $= 0.24\ \ (2\ dec\ \ pl)$ ☑	**1**
(d)(ii)	$= {}^{12}C_0(0.2)^0(0.8)^{12} + {}^{12}C_1(0.2)^1(0.8)^{11} + {}^{12}C_2(0.2)^2(0.8)^{10} + {}^{12}C_3(0.2)^3(0.8)^9$ ☑ $= 0.79456....$ $= 0.79\ \ (2\ dec\ \ pl)$ ☑	**2**
(e)	$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int_0^{\frac{3}{2}} \frac{2}{\sqrt{3^2-(2x)^2}}dx$ ☑ $= \frac{1}{2}\left[sin^{-1}\left(\frac{2x}{3}\right)\right]_0^{\frac{3}{2}}$ $= \frac{1}{2}\left[sin^{-1}(1) - sin^{-1}\left(0\right)\right]$ $= \frac{1}{2}\left(\frac{\pi}{2}\right)$ $= \frac{\pi}{4}$ ☑	**2**
(f)	x = number of students months = 12 $\frac{25}{12} = 2.08 \approx 3\ students$ Therefore $hence\ \ x = 2\times12+1 = 25 \qquad \therefore 25\ students$ ☑	**1**

Question 12 (15 marks) **Marks**

(a)	 • Finds all points of intersection with $y = f(x)$ or the coordinates of the new turning point ☑ . Graph must be a smooth graph. • Shows the new asymptote at $y = 2$ for $x \geq 0$ ☑	**2**
(b)	$\frac{dy}{y} = 2x\ dx$ $\int \frac{dy}{y} = \int 2x\ dx$ $\ln\lvert y\rvert = x^2 + C$ ☑ $sub\ x = 0,\ y = 2$ $\ln\lvert 2\rvert = 0^2 + C$ $\therefore c = \ln\lvert 2\rvert$ ☑ $hence\ \ \ln\lvert y\rvert = x^2 + \ln\lvert 2\rvert$ $y = e^{x^2 + \ln 2}$ (*note must simplify to next line to get full marks*) $y = 2e^{x^2}$ ☑	
(c)	Firstly select 4 spots, 10 for top row ie: $^{10}C_4$ Secondly select 2 of these 4 and shade ie: $^{4}C_2$ ☑ Finally, choose other 6 available spots with 4 more boxes ie: $^{6}C_4$ $\therefore\ ^{10}C_4 \times {}^{4}C_2 \times {}^{6}C_4 = 18900\ ways$ ☑ *Alternate method* : Firstly select 2 spots, 10 for top row ie: $^{10}C_2$ Secondly select 2 of these 4 and shade ie: $^{8}C_2$ ☑ Finally, choose other 6 available spots with 4 more boxes ie: $^{6}C_4$ $\therefore\ ^{10}C_2 \times {}^{8}C_2 \times {}^{6}C_4 = 18900\ ways$ ☑	**2**

(d)	Let $\sqrt{6}\sin x + \sqrt{2}\cos x = R\sin(x+\alpha)$ Then $\tan\alpha = \dfrac{\sqrt{2}}{\sqrt{6}} = \dfrac{1}{\sqrt{3}}$ and $R^2 = (\sqrt{6})^2 + (\sqrt{2})^2 = 8$ $\alpha = \arctan\left(\dfrac{1}{\sqrt{3}}\right)$ and $R^2 = 2+6 = 8$ $\alpha = \dfrac{\pi}{6}$ and $R = \sqrt{8}$ $\sqrt{6}\sin x + \sqrt{2}\cos x = \sqrt{8}\sin\left(x+\frac{\pi}{6}\right)$ ✓ $\sqrt{6}\sin x + \sqrt{2}\cos x = 2$ $\therefore \sqrt{8}\sin\left(x+\frac{\pi}{6}\right) = 2$ $\sin\left(x+\frac{\pi}{6}\right) = \dfrac{2}{\sqrt{8}} = \dfrac{2}{2\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ $x+\frac{\pi}{6} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$ ….. ✓ $x = \frac{\pi}{4}-\frac{\pi}{6}, \frac{3\pi}{4}-\frac{\pi}{6}, \frac{9\pi}{4}-\frac{\pi}{6}, \frac{11\pi}{4}-\frac{\pi}{6}$ …. $x = \dfrac{\pi}{12}, \dfrac{7\pi}{12}, \dfrac{25\pi}{12}, \dfrac{31\pi}{12}$ …. For required domain of $0 \le x \le 2\pi$ $x = \frac{\pi}{12}, \frac{7\pi}{12}$ ✓	**3**
(e)	Show that $9^n - 4^n$ is divisible by 5 for $n \ge 1$ Show that it is true for $n = 1$ $9^1 - 4^1 = 9 - 4 = 5 = 5 \times 1$ $\therefore$ true for $n = 1$ ✓ Assume that it is true for $n = k$ i.e. that $9^k - 4^k = 5p$ where p is an integer Show that it is true for $n = k+1$ i.e. that $9^{k+1} - 4^{k+1} = 5q$ where q is an integer LHS $= 9^{k+1} - 4^{k+1}$ $= 9 \times 9^k - 4 \times 4^k$ $= 9 \times 9^k - 4 \times 4^k - 5 \times 4^k + 5 \times 4^k$ $= 9 \times 9^k - 9 \times 4^k + 5 \times 4^k$ $= 9(9^k - 4^k) + 5 \times 4^k$ using assumption ✓ $= 9 \times 5p + 5 \times 4^k$ $= 5(9p + 4^k)$ $= 5q$ since $9p + 4^k$ is an integer, as p and k are integers ✓ $\therefore$ if true for $n = k$, it is also true for $n = k+1$ But true for $n = 1$, so by induction is true for all integers $n \ge 1$	**3**

(f)	$\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}, \quad \underset{\sim}{v} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$ $\text{proj}_{\underset{\sim}{u}}\underset{\sim}{v} = \frac{\underset{\sim}{v} \bullet \underset{\sim}{u}}{\underset{\sim}{u} \bullet \underset{\sim}{u}}\underset{\sim}{u}$ $= \frac{-4+12}{4+9}\underset{\sim}{u}$ ☑ $= \frac{8}{13}\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$ $= \frac{16}{13i} + \frac{24}{13}\underset{\sim}{j}$ ☑ OR $\frac{8}{13}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$	**2**

Question 13 (15 marks) **Marks**

(a)(i)	$\tan 4\theta - \tan\theta$ $= \dfrac{\sin 4\theta}{\cos 4\theta} - \dfrac{\sin\theta}{\cos\theta}$ ☑ $= \dfrac{\sin 4\theta\cos\theta - \sin\theta\cos 4\theta}{\cos 4\theta\cos\theta}$ $= \dfrac{\sin(4\theta-\theta)}{\cos 4\theta\cos\theta}$ ☑ $= \dfrac{\sin 3\theta}{\cos 4\theta\cos\theta}$	**2**
(a)(ii)	$\tan 4\theta = \tan\theta$ $\tan 4\theta - \tan\theta = 0$ $\dfrac{\sin 3\theta}{\cos 4\theta\cos\theta} = 0$ using result from (ii) $\sin 3\theta = 0 \quad 0 \le x \le \pi$ ☑ $3\theta = 0, \pi, 2\pi, 3\pi$ $\theta = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \pi$ ☑	**2**
(b)(i)	$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$ $\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$ ☑ $\overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{b}$ $\overrightarrow{BD} = \dfrac{1}{3}\left(\underset{\sim}{c} - \underset{\sim}{b}\right)$ $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$ $= \left(\underset{\sim}{b} - \underset{\sim}{a}\right) + \dfrac{1}{3}\left(\underset{\sim}{c} - \underset{\sim}{b}\right)$ $\overrightarrow{AD} = -\underset{\sim}{a} + \dfrac{2}{3}\underset{\sim}{b} + \dfrac{1}{3}\underset{\sim}{c}$ ☑ $\overrightarrow{AM} = \dfrac{1}{2}\overrightarrow{AD}$ $\therefore \overrightarrow{AM} = -\dfrac{1}{2}\underset{\sim}{a} + \dfrac{1}{3}\underset{\sim}{b} + \dfrac{1}{6}\underset{\sim}{c}$ ☑	**3**

(b)(ii)	$\overrightarrow{AX} = \frac{1}{4}\left(\underset{\sim}{c} - \underset{\sim}{a}\right)$ $\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM}$ $= \left(\underset{\sim}{a} - \underset{\sim}{b}\right) + \left(-\frac{1}{2}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b} + \frac{1}{6}\underset{\sim}{c}\right)$ ☑ $\overrightarrow{BM} = \frac{1}{2}\underset{\sim}{a} - \frac{2}{3}\underset{\sim}{b} + \frac{1}{6}\underset{\sim}{c}$ $\overrightarrow{MX} = \overrightarrow{MA} + \overrightarrow{AX}$ $= \left(\frac{1}{2}\underset{\sim}{a} - \frac{1}{3}\underset{\sim}{b} - \frac{1}{6}\underset{\sim}{c}\right) + \frac{1}{4}\left(\underset{\sim}{c} - \underset{\sim}{a}\right)$ $= \frac{1}{4}\underset{\sim}{a} - \frac{1}{3}\underset{\sim}{b} + \frac{1}{12}\underset{\sim}{c}$ $= \frac{1}{2}\left(\frac{1}{2}\underset{\sim}{a} - \frac{2}{3}\underset{\sim}{b} + \frac{1}{6}\underset{\sim}{c}\right)$ ☑ $\therefore \overrightarrow{MX} = \frac{1}{2}\overrightarrow{BM}$ Therefore $BM : MX = 2 : 1$. So MX and BM are in the same direction and have a common point, ☑ So B, M and X are collinear.	**3**
(c)(i)	$N = 70 + Ae^{0\cdot 1t}$ (or $Ae^{0\cdot 1t} = N - 70$) $\frac{dN}{dt} = 0.1 \times Ae^{0\cdot 1t}$ $= 0.1(N - 70)$ ☑	**1**
(c)(ii)	When $t = 0$ then $N = 100$ $100 = 70 + Ae^{0\cdot 1\times 0}$ $A = 30$ ☑ We need to find t when $N = 190$ $190 = 70 + 30e^{0\cdot 1t}$ $e^{0\cdot 1t} = \frac{120}{30} = 4$ $0.1t = \ln(4)$ $t = 10 \times \ln(4)$ $= 13.8629\ldots$ years ☑ $\therefore$ It will 2034 when the population size exceeds 190.	**2**

(d)	$y=\sin^{-1}\left(x+\frac{\pi}{2}\right)+\frac{\pi}{2}$ $\frac{dy}{dx}=\frac{1}{\sqrt{1-\left(x+\frac{\pi}{2}\right)^2}}$ ☑ $\quad x=-\frac{\pi}{2} \Rightarrow \begin{cases} y=\frac{\pi}{2} \\ \frac{dy}{dx}=1 \end{cases}$ Normal at $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ has gradient -1 and equation $y-\frac{\pi}{2}=-1\left(x+\frac{\pi}{2}\right)$ $y=-x$ ☑ hence passes through the origin.	**2**

Question 14 (15 marks) — Marks

(a)(i)

$x = Vt\cos\theta$, $y = Vt\sin\theta - 5t^2$

Let $t = 2,\ x = 15$:

$15 = V \times 2 \times \cos\theta$ ☑

$\cos\theta = \dfrac{15}{2V}$ (1)

Let $t = 2,\ y = 5$

$5 = V \times 2 \times \sin\theta - 5(2)^2$

$5 = 2V\sin\theta - 20$

$2V\sin\theta = 25$

$\sin\theta = \dfrac{25}{2V}$ (2) ☑

$(2) \div (1)$: $\tan\theta = \dfrac{25}{15}$

$\theta = \tan^{-1}\left(\dfrac{5}{3}\right)$

$= 59.036243$

$= 59°2'$

sub in (1): $\cos 59°2' = \dfrac{15}{2V}$

$V = \dfrac{15}{2\cos 59°2'}$

$= 14.577379$

$= 14.58\ \text{ms}^{-1}$ (2 dp) ☑

Marks: 5

Let $x = 10$

Since the horizontal velocity is constant, $t = \frac{10}{15} \times 2 = \frac{4}{3}$ seconds.

$y = 14.58 \times \dfrac{4}{3} \times \sin 59°2' - 5\left(\dfrac{4}{3}\right)^2$

$= 7.77777\ldots$

$= 7.78$ metres (2 decimal places)

$d_1 = 7.78 - 5 = 2.78$ metres. ☑

At impact $y = 0$

$\therefore\ 14.58t\sin 59°2' - 5t^2 = 0$

$t(14.58\sin 59°2' - 5t) = 0$

$t \neq 0,\quad t = \dfrac{14.58\sin 59°2'}{5}$

$= 2.5$ seconds

$\therefore\ 15 + d_2 = 14.58 \times 2.5 \times \cos 59°2'$

$= 18.75$

$d_2 = 3.75$ metres

$d_2 - d_1 = 3.75 - 2.78 = 0.97$ metres, so d_2 is less than 1 metre more than d_1. ☑

(b)(i)	$t=0, N=500. t=1, N=1500. \dfrac{dN}{dt}=kN(10000-N)$ (i) $\dfrac{10000}{N(10000-N)}=\dfrac{1}{N}+\dfrac{1}{10000-N}$ $\dfrac{dN}{dt}=kN(10000-N)$ $\dfrac{dN}{kN(10000-N)}=dt$ $t=\displaystyle\int\dfrac{dN}{kN(10000-N)}$ $=\dfrac{1}{10000k}\displaystyle\int\left(\dfrac{1}{N}+\dfrac{1}{10000-N}\right)dN$ $=\dfrac{1}{10000k}(\ln N-\ln(10000-N))+C$ ✓ $t=0, N=500:\quad 0=\dfrac{1}{10000k}\ln\left(\dfrac{500}{10000-500}\right)+C$ $C=\dfrac{-1}{10000k}\ln\left(\dfrac{1}{19}\right)=\dfrac{1}{10000k}\ln 19$ $t=\dfrac{1}{10000k}\ln\left(\dfrac{N}{10000-N}\right)+\dfrac{1}{10000k}\ln 19$ $10000kt=\ln\left(\dfrac{19N}{10000-N}\right)$ $\left(\dfrac{19N}{10000-N}\right)=e^{10000kt}$ ✓ $19N=10000e^{10000kt}-Ne^{10000kt}$ $N\left(19+e^{10000kt}\right)=10000e^{10000kt}$ $N=\dfrac{=10000e^{10000kt}}{19+e^{10000kt}}$ $N=\dfrac{10000}{1+19e^{-10000kt}}$ ✓ OR (i) $\dfrac{10000}{N(10000-N)}=\dfrac{1}{N}+\dfrac{1}{10000-N}$ $\dfrac{dN}{dt}=kN(10000-N)$ $\dfrac{dN}{kN(10000-N)}=dt$ $\displaystyle\int_0^t dt=\int_{500}^{N}\dfrac{dN}{kN(10000-N)}$ $t=\dfrac{1}{10000k}\displaystyle\int_{500}^{N}\left(\dfrac{1}{N}+\dfrac{1}{10000-N}\right)dN$ $=\dfrac{1}{10000k}\left[\ln N-\ln(10000-N)\right]_{500}^{N}$ $=\dfrac{1}{10000k}\left(\ln\left(\dfrac{N}{10000-N}\right)-\ln\left(\dfrac{500}{9500}\right)\right)$ $=\dfrac{1}{10000k}\left(\ln\left(\dfrac{N}{10000-N}\right)-\ln\left(\dfrac{1}{19}\right)\right)$ $=\dfrac{1}{10000k}\left(\ln\left(\dfrac{19N}{10000-N}\right)\right)$ $10000kt=\ln\left(\dfrac{19N}{10000-N}\right)$ $\left(\dfrac{19N}{10000-N}\right)=e^{10000kt}$ $19N=10000e^{10000kt}-Ne^{10000kt}$ $N\left(19+e^{10000kt}\right)=10000e^{10000kt}$ $N=\dfrac{=10000e^{10000kt}}{19+e^{10000kt}}$ $N=\dfrac{10000}{1+19e^{-10000kt}}$	**3**
(b)(ii)	$N=\dfrac{10\,000}{1+19e^{-10000kt}},\quad k=\dfrac{1}{10000}\ln\left(\dfrac{57}{17}\right)$ or $-10000k=\ln\dfrac{17}{57}$ $N=7000:\quad 7000=\dfrac{10\,000}{1+19e^{-10000kt}}$ $1+19e^{-10000kt}=\dfrac{10}{7}$ $19e^{-10000kt}=\dfrac{3}{7}$ $e^{-10000kt}=\dfrac{3}{133}$ ✓ $-10000kt=\ln\left(\dfrac{3}{133}\right)$ $t\ln\left(\dfrac{17}{57}\right)=\ln\left(\dfrac{3}{133}\right)$ $t=\dfrac{\ln\left(\dfrac{3}{133}\right)}{\ln\left(\dfrac{17}{57}\right)}$ $t=3.1341$ years ≈ 38 months ✓	**2**

(c)(i)	99.7% certainty equates to 3 standard deviations either side of the mean, p. ☑ This means that the maximum error of 0.001 has to equal 3 standard deviations. $3\sigma = 0.001$ $\sigma = \frac{0.001}{3} = 3.\dot{3} \times 10^{-4}$ ☑ Using the estimate of $p = 0.81$ we have: $3.\dot{3} \times 10^{-4} = \sqrt{\frac{0.81 \times 0.19}{n}}$ $1.\dot{1} \times 10^{-7} = \frac{0.1539}{n}$ $n = \frac{0.1539}{1.\dot{1} \times 10^{-7}} = 1\ 385\ 100$ The experiment must be repeated at least 1 385 100 times to be 99.7% ☑ certain of knowing the value of p with an error of less than one-thousandth.	**3**
(c)(ii)	$\mu_{\hat{p}} = \hat{p}$ $= \frac{1\ 615\ 580}{2\ 000\ 000}$ $= 0.80779$ $\sigma_{\hat{p}} = \sqrt{\frac{0.80779 \times (1 - 0.80779)}{10\ 000}}$ $= 0.00394037201$ ☑ $z = \frac{0.81 - 0.80779}{0.00394037201}$ $= 0.5608607496$ $= 0.56$ (2 dp) $P(Z > 0.56) = 1 - P(Z < 0.56)$ $= 1 - 0.7123$ $= 0.2877$ $= 28.8\%$ (1 dp) ☑	**2**

Alternate Solution to Q14(c)(i)

Q14(c)(i) $X \sim Bin(n, p) \qquad p \approx 0.81$

$$P\left(\frac{-p}{1000} < \hat{p} - p < \frac{p}{1000}\right) = 0.997$$

$$\hat{p} = \frac{X}{n} \sim N\left(p, \sigma^2 = \frac{p(1-p)}{n}\right)$$

$$\therefore P\left(\frac{-p}{1000\sqrt{\frac{p(1-p)}{n}}} < \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} < \frac{p}{1000\sqrt{\frac{p(1-p)}{n}}}\right) = 0.997$$

$$\therefore \frac{p}{1000\sqrt{\frac{p(1-p)}{n}}} = 3 \qquad (\text{or } 2.97)$$

let $p = 0.81$

$$\frac{0.81}{1000\sqrt{\frac{0.81 \times 0.19}{n}}} = 3$$

$$\frac{0.81}{3000} = \sqrt{\frac{0.1539}{n}}$$

$$\left(\frac{3000}{0.81}\right)^2 = \frac{n}{0.1539}$$

$$n \doteq 2{,}111{,}111$$